Generalized flag geometries and manifolds associated to short Z-graded Lie algebras in arbitrary dimension.

Julien Chenal

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Chenal, Julien [a],

[a] *Institut Elie Cartan Nancy(IECN), Nancy-Université, CNRS, INRIA, Boulevard des Aiguillettes B.P. 239 F-54506 Vandoeuvre-lès-Nancy*

Abstract

The object of this note is to define the generalized flag geometry of a graded Lie algebra which corresponds to the generalized projective geometry in the case of 3-gradings. Then we construct a structure of manifold on this generalized flag geometry. This result generalizes a result known for 3-graded Lie algebras to the more general case of $(2k+1)$-graded Lie algebras.

Résumé

Géométries de drapeaux généralisées et variétés associées aux algèbres de Lie graduées en dimension quelconque.

L'objet de cette note est de définir la géométrie de drapeaux généralisée d'une algèbre de Lie graduée, qui correspond à la géométrie projective généralisée dans le cas des 3-graduations, puis de construire une structure de variété différentielle sur cette géométrie. Ce résultat généralise au cas des $(2k+1)$-graduations un résultat déjà connu pour les 3-graduations.

Version française abrégée

Dans une première partie, nous considérons le groupe projectif élémentaire $G := PE(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$ d'une algèbre de Lie graduée $\mathfrak{g} = \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{-k}$. C'est le groupe engendré par les deux groupes $U^+ := \left\{ e^{\mathrm{ad}(x)},\ x \in \bigoplus_{i=1}^{k} \mathfrak{g}_i \right\}$ et $U^- := \left\{ e^{\mathrm{ad}(x)},\ x \in \bigoplus_{i=1}^{k} \mathfrak{g}_{-i} \right\}$ (cf. [3], [5]) ; puis on considère trois espaces homogènes : $X^+ := PE(\mathfrak{g})/P^-,\ X^- := PE(\mathfrak{g})/P^+$, où pour $\sigma \in \{+,-\}$, $P^\sigma := U^\sigma H$ avec H le sous-groupe des éléments du groupe projectif élémentaire qui préservent la graduation, et $M := PE(\mathfrak{g})/H$.

Email address: `chenal@iecn.u-nancy.fr` (Chenal, Julien).

Preprint submitted to the Académie des sciences *13 novembre 2008*

Alors M s'injecte dans $X^+ \times X^-$ et plus précisément, M est isomorphe à l'orbite du point de base (P^-, P^+) dans $X^+ \times X^-$ sous l'action du groupe projectif élémentaire. Enfin, sur $X^+ \times X^-$, on définit une relation de transversalité : le couple (x, y) où $x \in X^+$ et $y \in X^-$, est dit *transverse* si $(x, y) \in M$. Alors (X^+, X^-), avec la relation de transversalité forme ce que nous appellons la *géométrie de drapeaux généralisée* associée à la graduation de $\mathfrak{g}$.

Dans une deuxième partie, nous utilisons les filtrations de $\mathfrak{g}$, i.e les drapeaux de sous-espaces de $\mathfrak{g}$ de la forme $0 = \mathfrak{n}_{k+1} \subset \mathfrak{n}_k \subset \mathfrak{n}_{k-1} \subset \cdots \subset \mathfrak{n}_0 \subset \mathfrak{n}_{-1} \subset \cdots \subset \mathfrak{n}_{-k+1} \subset \mathfrak{n}_{-k} = \mathfrak{g}$ avec $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$, pour obtenir une réalisation géométrique des espaces X^+ et X^-. On obtient X^+ et X^- comme les orbites sous l'action du groupe projectif élémentaire des filtrations associées à la graduation de $\mathfrak{g}$: $X^+ \cong G.\mathfrak{n}^-$ et $X^- \cong G.\mathfrak{n}^+$ où

$$\mathfrak{n}^+ := (\mathfrak{g}_k \subset \mathfrak{g}_k \oplus \mathfrak{g}_{k-1} \subset \cdots \subset \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_0 \subset \cdots \subset \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_{-k+1}) \text{ et}$$
$$\mathfrak{n}^- := (\mathfrak{g}_{-k} \subset \mathfrak{g}_{-k} \oplus \mathfrak{g}_{-k+1} \subset \cdots \subset \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \subset \cdots \subset \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{k-1}).$$

Enfin, dans une dernière partie, nous utilisons la notion de calcul différentiel sur un corps topologique non discret (cf.[1]) pour construire une structure de variété différentielle sur X^+. Cette construction généralise celle de [6] pour les algèbres de Lie 3-graduées. Pour cela, on suppose que $\mathfrak{n}_1^+$ et $\mathfrak{n}_1^-$ sont des algèbres de Lie topologiques et nous définissons les *opérateurs de Bergman généralisés*, qui généralisent la notion d'opérateur de Bergman dans une paire de Jordan (cf. [2]) : pour $x \in \mathfrak{n}_1^+$, $w \in \mathfrak{n}_1^-$, et $1 \le j \le k$, on pose :

$$B^+(x, w)_j = \mathrm{pr}_{\mathfrak{n}_j^+} \circ (e^{-\mathrm{ad}(x)} e^{-\mathrm{ad}(w)}) \circ \iota_{\mathfrak{n}_j^+} \quad \text{et } B^-(w, x)_j = \mathrm{pr}_{\mathfrak{n}_j^-} \circ (e^{\mathrm{ad}(w)} e^{\mathrm{ad}(x)}) \circ \iota_{\mathfrak{n}_j^-}$$

où $\mathrm{pr}_{\mathfrak{n}_j^\sigma}$ est la projection de $\mathfrak{g}$ sur $\mathfrak{n}_j^\sigma$ et $\iota_{\mathfrak{n}_j^\sigma}$ l'inclusion de $\mathfrak{n}_j^\sigma$ dans $\mathfrak{g}$, pour $\sigma \in \{+, -\}$.
Alors, nous montrons que sous des hypothèses (H1) et (H2) formulées ci-dessous, il existe sur X^+ une structure de variété différentielle (Theorem 3.2).

1. Definition of generalized flag geometries

In the sequel, $\sigma \in \{+, -\}$ and we assume that $\mathbb{K}$ is a commutative ring in which the integers are invertible.

A $\mathbb{Z}$-*graded Lie algebra* (over $\mathbb{K}$) is a Lie algebra over $\mathbb{K}$ of the form $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$, with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
A $(2k+1)$-*graded Lie algebra* is a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ such that $\mathfrak{g}_j = 0$ if $|j| > k$.
Let $\mathfrak{g}$ be a $(2k+1)$-graded Lie algebra. The map $D : \mathfrak{g} \to \mathfrak{g}$ defined by $DX = iX$ if $X \in \mathfrak{g}_i$ is a derivation of $\mathfrak{g}$ called the *characteristic derivation* of the grading. If $D = \mathrm{ad}(E)$, $E \in \mathfrak{g}$, D is said an *inner derivation* and E is called an *Euler operator*. Finite dimensional real simple graded Lie algebras have been classified in [7].
For $x \in V^+ := \bigoplus_{i=1}^{k} \mathfrak{g}_i$, or $x \in V^- := \bigoplus_{i=1}^{k} \mathfrak{g}_{-i}$, we define the operator

$$e^{\mathrm{ad}(x)} := \sum_{j=0}^{2k} \frac{1}{j!} \mathrm{ad}(x)^j.$$

By our assumption on $\mathbb{K}$, $e^{\mathrm{ad}(x)}$ is well-defined and is an automorphism of $\mathfrak{g}$. We define the groups generated by these operators,

$$U^+ := U^+(D) := \left\{ e^{\mathrm{ad}(x)},\ x \in V^+ \right\}, \quad U^- := U^-(D) := \left\{ e^{\mathrm{ad}(x)},\ x \in V^- \right\}$$

and the group generated by U^+ and U^-:

Definition 1.1 *The group* $G := PE(\mathfrak{g}, D) := < U^+, U^- > \subset Aut(\mathfrak{g})$ *is called the* projective elementary group of $(\mathfrak{g}, D)$ (cf. [3] and [5]).

We define also the *automorphism group of* $(\mathfrak{g}, D)$ by

$$\mathrm{Aut}(\mathfrak{g}, D) := \{g \in \mathrm{Aut}(\mathfrak{g}),\ g \circ D = D \circ g\}$$

and the subgroups

$$H := G(D) \cap \mathrm{Aut}(\mathfrak{g}, D) \quad \text{and} \quad P^\sigma := HU^\sigma = U^\sigma H.$$

The elements of H commute with the derivation D, so they preserve the grading, and normalize U^σ so P^+ and P^- are subgroups of G. Finally, we define homogeneous spaces:

$$X^+ := G/P^-, \quad X^- := G/P^+ \quad \text{and} \quad M := G/H$$

We note (o^+, o^-) the base point (P^-, P^+) of $X^+ \times X^-$.

Definition 1.2 *On* $X^+ \times X^-$*, we define the* relation of transversality*: for* $x \in X^+$ *and* $y \in X^-$*,* x *and* y *are said* transversal *if and only if* (x, y) *are conjugate under* G *to the base point. If* x *and* y *are transversal, we will write* $x \top y$*. We define also* $(X^+ \times X^-)^\top := \{(x, y) \in X^+ \times X^-,\ x \top y\}$ *and for* $x \in X^\sigma$*,* $x^\top := \{y \in X^{-\sigma}, x \top y\}$*.*
The data $(X^+, X^-, \top)$ *is called a* generalized flag geometry of k-graded type.

Then the relation of transversality can be defined using the space M thanks to the following:
Lemma 1.3

i) If $x = g \cdot o^+ \in X^+$ *and* $y = g' \cdot o^- \in X^-$*, then* x *and* y *are transversal if and only if there exists* $z \in V^+$ *such that* $x = g' e^{ad(z)} \cdot o^+$*.*
ii) We have $P^+ \cap P^- = H$*. Hence* $V^+ \subset X^+$ *via* $x \mapsto e^{ad(x)} \cdot o^+$ *and* $(o^-)^\top \cong V^+$*.*
iii) The space M *is isomorphic to* $(X^+ \times X^-)^\top$*, i.e* H *is the stabilizer of the base point* (o^+, o^-)*.*

2. Geometric realization and filtrations of a Lie algebra

In this section, we will give a geometric realization of the homogeneous spaces X^+ and X^- together with the transversality relation. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. A $(2k+1)$-filtration of $\mathfrak{g}$ is a flag of subspaces

$$0 = \mathfrak{n}_{k+1} \subset \mathfrak{n}_k \subset \mathfrak{n}_{k-1} \subset \cdots \subset \mathfrak{n}_0 \subset \mathfrak{n}_{-1} \subset \cdots \subset \mathfrak{n}_{-k+1} \subset \mathfrak{n}_{-k} = \mathfrak{g} \ \text{ with } [\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}.$$

Such a flag will be denoted by $\mathfrak{n} = (\mathfrak{n}_k \subset \cdots \subset \mathfrak{n}_{-k+1})$, and the *space of* $(2k+1)$*-filtrations of* $\mathfrak{g}$ will be denoted by $\widetilde{\mathcal{F}}$. If $\mathfrak{n} \in \widetilde{\mathcal{F}}$, then $\mathrm{ad}(X)$ with $X \in \mathfrak{n}_1$ is nilpotent and hence the automorphism $e^{\mathrm{ad}(X)}$ is well-defined. We denote by

$$U(\mathfrak{n}) := e^{\mathrm{ad}(\mathfrak{n}_1)} = \{e^{\mathrm{ad}(X)},\ X \in \mathfrak{n}_1\} \subset \mathrm{Aut}(\mathfrak{g})$$

the corresponding group. Since $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$, the group $U(\mathfrak{n})$ preserves the filtration $\mathfrak{n}$.
To any $(2k+1)$-grading, $\mathfrak{g} = \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{-k}$ of $\mathfrak{g}$, with characteristic derivation D, we may associate two $(2k+1)$-filtrations of $\mathfrak{g}$, called the *associated plus- and minus-filtration*, defined by

$$\mathfrak{n}^+(D) := (\mathfrak{g}_k \subset \mathfrak{g}_k \oplus \mathfrak{g}_{k-1} \subset \cdots \subset \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_0 \subset \cdots \subset \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_{-k+1}) \text{ and}$$
$$\mathfrak{n}^-(D) := (\mathfrak{g}_{-k} \subset \mathfrak{g}_{-k} \oplus \mathfrak{g}_{-k+1} \subset \cdots \subset \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \subset \cdots \subset \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{k-1}).$$

A $(2k+1)$-filtration $\mathfrak{n}$ of $\mathfrak{g}$ is said *inner* if $\mathfrak{n} = \mathfrak{n}^+(\mathrm{ad}(E))$, $E \in \mathfrak{g}$. The *space of inner* $(2k+1)$*-filtrations of* $\mathfrak{g}$ will be denoted by $\mathcal{F}$.

Definition 2.1 *Let* $(\mathfrak{m}, \mathfrak{n})$ *be two* $(2k+1)$*-filtrations of* $\mathfrak{g}$*. We say that* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are* transversal *if, for* $-k+1 \leq i \leq k$*, we have* $\mathfrak{g} = \mathfrak{n}_i \oplus \mathfrak{m}_{-i+1}$*. If* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are transversal, we will write* $\mathfrak{m} \top \mathfrak{n}$*.*

By construction, the filtrations associated to a derivation D, $\mathfrak{n}^+(D)$ and $\mathfrak{n}^-(D)$ are transversal. Conversely, we will prove that any pair of transversal filtrations arises in this way. In this case, $X^\sigma \cong G \cdot \mathfrak{n}^{-\sigma}$, and the two notions of transversality (Definitions 1.2 and 2.1) are compatible:

Theorem 2.2 *There is an embedding* $X^+ \times X^- \hookrightarrow \mathcal{F} \times \mathcal{F}$ *which is G-equivariant and compatible with transversality. More precisely,*

i) the orbit of the filtration $\mathfrak{n}^\sigma$ *under the action of* G *is isomorphic to* $X^{-\sigma}$ *i.e* P^σ *is the stabilizer of the filtration* $\mathfrak{n}^\sigma$*:* $P^\sigma = \{g \in G,\ g.\mathfrak{n}^\sigma = \mathfrak{n}^\sigma\}$*. Hence* $H = \{g \in G,\ g.(\mathfrak{n}^+, \mathfrak{n}^-) = (\mathfrak{n}^+, \mathfrak{n}^-)\}$*.*
ii) Two inner filtrations are transversal if and only if they come from a grading of $\mathfrak{g}$*.*
iii) Let $\mathfrak{n} \in \mathcal{F}$*. Then* $e^{ad(\mathfrak{n}_1)}$ *acts simply transitively on* $\mathfrak{n}^\top := \{\mathfrak{m} \in \mathcal{F},\ \mathfrak{m} \top \mathfrak{n}\}$*.*
iv) We have an injection $\mathfrak{n}_1^+ \to X^+,\ x \mapsto e^{ad(x)} \cdot o^+$*. Then the map*

$$\mathfrak{n}_1^+ \times H \times \mathfrak{n}_1^- \to \Omega^+, \quad (v, h, w) \mapsto e^{ad(v)} h e^{ad(w)} \text{ is a bijection, where } \Omega^+ := \{g \in G,\ g.o^+ \in \mathfrak{n}_1^+\}.$$

Proof. Here, we will only prove ii) which is the key property to prove the other points. First, we prove that for $E \in \mathfrak{g}$ and $\mathfrak{n} = \mathfrak{n}^+(\mathrm{ad}(E))$, we have $e^{\mathrm{ad}(\mathfrak{n}_1)} E = E + \mathfrak{n}_1$.
Clearly, $e^{\mathrm{ad}(\mathfrak{n}_1)} E \subset E + \mathfrak{n}_1$.
In order to prove the other inclusion, let $X \in \mathfrak{n}_1$. We will prove by induction that for all $n \in \{3, \ldots, k+1\}$, there exists $Y^n \in \mathfrak{n}_1$ such that $e^{\mathrm{ad}(Y^n)} E = E + X + R^n$ where $R^n \in \mathfrak{n}_n$. If $n = k+1$, then $\mathfrak{n}_{k+1} = 0$ so $R^{k+1} = 0$ and $e^{\mathrm{ad}(Y^{k+1})} E = E + X$.
Since $X \in \mathfrak{n}_1$, we have $X = \sum_{i=1}^{k} X_i$ with $X_i \in \mathfrak{g}_i$. Let

$$Y^3 := \sum_{i=1}^{k} -\frac{1}{i} X_i.$$

Then $e^{\mathrm{ad}(Y^3)} E = E + X + R^3$, where $R^3 := \sum_{j=2}^{k} \frac{1}{j!} \mathrm{ad}(Y^3)^j E \in \mathfrak{n}_3$.
Now, let $n \in \{3, \ldots, k\}$. We assume that there exists $Y^n \in \mathfrak{n}_1$ such that $e^{\mathrm{ad}(Y^n)} E = E + X + R^n$ with $R^n \in \mathfrak{n}_n$. Since $R^n \in \mathfrak{n}_n$, we have $R^n = \sum_{i=n}^{k} R_i^n$ with $R_i^n \in \mathfrak{g}_i$. Let

$$Y^{n+1} := Y^n + \frac{1}{n} R_n^n \in \mathfrak{n}_1.$$

Then we have $e^{\mathrm{ad}(Y^{n+1})} E = E + [Y^n, E] + \frac{1}{n} [R_n^n, E] + \sum_{j=2}^{k} \frac{1}{j!} \mathrm{ad}(Y^n + \frac{1}{n} R_n^n)^j E.$

So $e^{\mathrm{ad}(Y^{n+1})} E = e^{\mathrm{ad}(Y^n)} E - R_n^n + \widetilde{R^n}$. Since the first term in $\widetilde{R^n}$ is $\left[Y^n, \left[\frac{1}{n} R_n^n, E\right]\right] = -[Y^n, R_n^n] \in \mathfrak{n}_{n+1}$, we have $\widetilde{R^n} \in \mathfrak{n}_{n+1}$. Hence $e^{\mathrm{ad}(Y^{n+1})}E = E + X + \sum_{i=n+1}^{k} R_i^n + \widetilde{R^n} = E + X + R^{n+1}$ with $R^{n+1} := \sum_{i=n+1}^{k} R_i^n + \widetilde{R^n} \in \mathfrak{n}_{n+1}$. Finally, there exists $Y := Y^{k+1} \in \mathfrak{n}_1$ such that $e^{\mathrm{ad}(Y)}E = E + X$. So $E + \mathfrak{n}_1 \subset e^{\mathrm{ad}(\mathfrak{n}_1)}E$ and then $e^{\mathrm{ad}(\mathfrak{n}_1)}E = E + \mathfrak{n}_1$.

Now let us prove ii). We have already remarked that $\mathfrak{n}^+(\mathrm{ad}(E))$ and $\mathfrak{n}^-(\mathrm{ad}(E))$ are transversal. In order to prove the converse, let $(\mathfrak{m}, \mathfrak{n})$ be two transversal inner filtrations. Since $\mathfrak{n}$ is inner, there exists $E' \in \mathfrak{g}$ such that $\mathfrak{n} = \mathfrak{n}^+(\mathrm{ad}(E'))$. Let $\mathfrak{g} = \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{-k}$ be the $(2k+1)$-grading of $\mathfrak{g}$ associated to $\mathrm{ad}(E')$. As $\mathfrak{m} \top \mathfrak{n}$, we have $\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{m}_0$, so there exists $Z' \in \mathfrak{n}_1$ such that $E' - Z' \in \mathfrak{m}_0$.
But there exists $Z \in \mathfrak{n}_1$ such that $e^{\mathrm{ad}(Z)}E' = E' - Z'$ since $e^{\mathrm{ad}(\mathfrak{n}_1)}E' = E' + \mathfrak{n}_1$. Then let

$$E := e^{\mathrm{ad}(Z)}E' = E' - Z' \in \mathfrak{m}_0.$$

Let us prove that $\mathfrak{n} = \mathfrak{n}^+(\mathrm{ad}(E))$ and $\mathfrak{m} = \mathfrak{n}^-(\mathrm{ad}(E))$. Indeed, since $Z \in \mathfrak{n}_1$, we have $\mathfrak{n}^+(\mathrm{ad}(E)) = \mathfrak{n}^+(\mathrm{ad}(e^{\mathrm{ad}(Z)}E')) = e^{\mathrm{ad}(Z)}\mathfrak{n}^+(\mathrm{ad}(E')) = e^{\mathrm{ad}(Z)}\mathfrak{n} = \mathfrak{n}$. So $\mathfrak{n} = \mathfrak{n}^+(\mathrm{ad}(E))$.
On the other hand, by construction $E \in \mathfrak{m}_0$, so the filtration $\mathfrak{m}$ is $\mathrm{ad}(E)$-stable. Moreover, $\mathfrak{n} = \mathfrak{n}^+(\mathrm{ad}(E))$ is also $\mathrm{ad}(E)$-stable. So $\mathfrak{g} = \mathfrak{n}_{-k+1} \oplus \mathfrak{m}_k$ and this decomposition is $\mathrm{ad}(E)$-stable. As $\mathfrak{n}_{-k+1} = \mathrm{Ker}(\mathrm{ad}(E) - k\mathrm{Id}) \oplus \cdots \oplus \mathrm{Ker}(\mathrm{ad}(E) + (k-1)\mathrm{Id})$, we have $\mathfrak{m}_k = \mathrm{Ker}(\mathrm{ad}(E) + k\mathrm{Id})$ i.e $\mathfrak{m}_k = \mathfrak{n}^-(\mathrm{ad}(E))_k$.
With the same argument, it follows that $\mathfrak{m}_{-i+1} = \mathfrak{n}^-(\mathrm{ad}(E))_{-i+1}$.
So $\mathfrak{m} = \mathfrak{n}^-(\mathrm{ad}(E))$ and finally, $\mathfrak{n} = \mathfrak{n}^+(\mathrm{ad}(E))$ and $\mathfrak{m} = \mathfrak{n}^-(\mathrm{ad}(E))$. □

3. Structure of manifold on X^+

Let $\mathfrak{g}$ be a $(2k+1)$-graded Lie algebra and $\mathfrak{n}^+$ and $\mathfrak{n}^-$ the plus- and minus-filtration of $\mathfrak{g}$. For $x \in \mathfrak{n}_1^+$ and $g \in \mathrm{Aut}(\mathfrak{g})$, we define the *denominators* and *co-denominators* of g, for $1 \leq j \leq k$ by:

$$d_g(x)_j = \mathrm{pr}_{\mathfrak{n}_j^+} \circ (e^{-\mathrm{ad}(x)}g^{-1}) \circ \iota_{\mathfrak{n}_j^+} \in \mathrm{End}(\mathfrak{n}_j^+) \quad \text{and } c_g(x)_j = \mathrm{pr}_{\mathfrak{n}_j^-} \circ (ge^{\mathrm{ad}(x)}) \circ \iota_{\mathfrak{n}_j^-} \in \mathrm{End}(\mathfrak{n}_j^-)$$

where $\mathrm{pr}_{\mathfrak{n}_j^\sigma}$ is the projection of $\mathfrak{g}$ onto $\mathfrak{n}_j^\sigma$ and $\iota_{\mathfrak{n}_j^\sigma}$ is the inclusion of $\mathfrak{n}_j^\sigma$ into $\mathfrak{g}$. Then, for all $1 \leq j \leq k$, $\mathfrak{n}_1^+ \to \mathrm{End}(\mathfrak{n}_j^+)$, $x \mapsto d_g(x)_j$ and $\mathfrak{n}_1^+ \to \mathrm{End}(\mathfrak{n}_j^-)$, $x \mapsto c_g(x)_j$ are polynomial maps.

Definition 3.1 *If $x \in \mathfrak{n}_1^+$ and $w \in \mathfrak{n}_1^-$, let $B^+(x,w)_j := d_{e^{ad(w)}}(x)_j$ and $B^-(w,x)_j := c_{e^{ad(w)}}(x)_j$, for $1 \leq j \leq k$. These operators are called the* generalized Bergman operators.

These operators can be seen as a generalization of the Bergman operator in Jordan Pairs (cf. [2] and [5]). Now, we assume that $\mathbb{K}$ is a non discrete topological field and $\mathfrak{n}_1^+$ and $\mathfrak{n}_1^-$ are topological Lie algebras over $\mathbb{K}$. Then under the following conditions on $(\mathfrak{n}_1^+, \mathfrak{n}_1^-)$, we can build on X^+ a structure of manifold modeled on $\mathfrak{n}_1^+$ (cf. [1] for the definition of differential calculus over a general field and [6] for the case of a 3-graded Lie algebra). It should be remarked that we do not consider a topology on $\mathfrak{g}_0$.

(H1) First, we assume that the maps

$$\begin{array}{lll} \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j} & \text{and} & \mathfrak{g}_k \times \mathfrak{g}_l \times \mathfrak{g}_m \to \mathfrak{g}_{k+l+m} \\ (x,y) \mapsto [x,y] & & (x,y,z) \mapsto [[x,y],z] \end{array}$$

when $i + j \neq 0$ and $k + l + m \neq 0$ are continuous.

(H2) In addition, we assume that the set denoted by $(\mathfrak{n}^+ \times \mathfrak{n}^-)^\times$ and defined by

$$(\mathfrak{n}^+ \times \mathfrak{n}^-)^\times := \left\{(x,w) \in \mathfrak{n}_1^+ \times \mathfrak{n}_1^-,\ B^+(x,w)_j \in \mathrm{GL}(\mathfrak{n}_j^+),\ B^-(w,x)_j \in \mathrm{GL}(\mathfrak{n}_j^-), \forall\, 1 \le j \le k\right\}$$

is open in $\mathfrak{n}_1^+ \times \mathfrak{n}_1^-$ and the map

$$\begin{array}{ccc} (\mathfrak{n}^+ \times \mathfrak{n}^-)^\times \times \mathfrak{n}_1^+ \times \mathfrak{n}_1^- & \to & \mathfrak{n}_1^+ \times \mathfrak{n}_1^- \\ ((x,w),(u,v)) & \mapsto & (B^+(x,w)_1^{-1}u, B^-(w,x)_1^{-1}v) \end{array} \quad \text{is continuous.}$$

Theorem 3.2 (Manifold structure of X^+)

i) The map $(\mathfrak{n}^+ \times \mathfrak{n}^-)^\times \to \mathfrak{n}_1^+ \times \mathfrak{n}_1^-, (x,w) \mapsto (e^{ad(x)} \cdot w,\ e^{ad(w)} \cdot x)$ is of class C^∞.

ii) For all $g \in G$ and for all $x \in \mathfrak{n}_1^+$, the action of $d_g(x)$ on $\mathfrak{n}_1^+$ is continuous.

iii) There exists on X^+ a structure of a smooth manifold modeled on $\mathfrak{n}_1^+$, uniquely defined by the fact that $\mathcal{A} = \left\{(g(\mathfrak{n}_1^+), \varphi_g),\ g \in G\right\}$ is an atlas of X^+, where $\varphi_g : g(\mathfrak{n}_1^+) \to \mathfrak{n}_1^+, g \cdot x \mapsto x$ for $g \in G$.

iv) The group G acts by diffeomorphisms on X^+.

v) The set $M \subset X^+ \times X^-$ is open.

The proof of this result will be given elsewhere. It relies on Theorem 2.2.
Let us add the remark that there is reasonable topology neither on G nor on H. So our proof cannot use the classical arguments establishing a manifold structure on the quotient of a Lie group by some (closed) subgroup. Indeed, if $\mathfrak{g}$ is real or complex, finite dimensional, and semisimple, it is enough to show that P^+ is closed to have a strucure of manifold on X^+. But, in Theorem 3.2, there is no assumption on the dimension of $\mathfrak{g}$ and $\mathbb{K}$ is not only $\mathbb{R}$ or $\mathbb{C}$.
Finally, concerning the condition (H2), if $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and if $\mathfrak{n}_1^+$ and $\mathfrak{n}_1^-$ are Banach spaces (of finite dimension or not), then under the condition (H1), the generalized Bergman operators $B^+(x,w)_1$ and $B^-(w,x)_1$ belong to the Banach algebra $L(\mathfrak{n}_1^+)$ or $L(\mathfrak{n}_1^-)$ of the continuous linear operators on $\mathfrak{n}_1^+$ or $\mathfrak{n}_1^-$ and inversion in these spaces is continuous (cf. [4]), so (H2) is automatically true.

I am grateful to the referee for his helpful remarks.

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